MATLAB Course
November-December 2006

Chapter 4: Optimization

```
>> help fminunc

 FMINUNC  Finds the minimum of a function of several variables.
   X=FMINUNC(FUN,X0) starts at X0 and finds a minimum X of the function
   FUN. FUN accepts input X and returns a scalar function value F evaluated
   at X. X0 can be a scalar, vector or matrix.

   X=FMINUNC(FUN,X0,OPTIONS)  minimizes with the default optimization
   parameters replaced by values in the structure OPTIONS, an argument
   created with the OPTIMSET function.  See OPTIMSET for details.  Used
   options are Display, TolX, TolFun, DerivativeCheck, Diagnostics, GradObj,
   HessPattern, LineSearchType, Hessian, HessMult, HessUpdate, MaxFunEvals,
   MaxIter, DiffMinChange and DiffMaxChange, LargeScale, MaxPCGIter,
   PrecondBandWidth, TolPCG, TypicalX. Use the GradObj option to specify that
   FUN also returns a second output argument G that is the partial
   derivatives of the function df/dX, at the point X. Use the Hessian option
   to specify that FUN also returns a third output argument H that
   is the 2nd partial derivatives of the function (the Hessian) at the
   point X.  The Hessian is only used by the large-scale method, not the
   line-search method.

   X=FMINUNC(FUN,X0,OPTIONS,P1,P2,...) passes the problem-dependent
   parameters P1,P2,... directly to the function FUN, e.g. FUN would be
   called using feval as in: feval(FUN,X,P1,P2,...).
   Pass an empty matrix for OPTIONS to use the default values.

   [X,FVAL]=FMINUNC(FUN,X0,...) returns the value of the objective
   function FUN at the solution X.

   [X,FVAL,EXITFLAG]=FMINUNC(FUN,X0,...) returns a string EXITFLAG that
   describes the exit condition of FMINUNC.
   If EXITFLAG is:
     > 0 then FMINUNC converged to a solution X.
     0   then the maximum number of function evaluations was reached.
     < 0 then FMINUNC did not converge to a solution.

     [X,FVAL,EXITFLAG,OUTPUT]=FMINUNC(FUN,X0,...) returns a structure
     OUTPUT with the number of iterations taken in OUTPUT.iterations, the number of
     function evaluations in OUTPUT.funcCount, the algorithm used in
```

OUTPUT.algorithm, the number of CG iterations (if used) in OUTPUT.cgiterations, and the first-order optimality (if used) in OUTPUT.firstorderopt.

[X,FVAL,EXITFLAG,OUTPUT,GRAD]=FMINUNC(FUN,X0,...) returns the value of the gradient of FUN at the solution X.

[X,FVAL,EXITFLAG,OUTPUT,GRAD,HESSIAN]=FMINUNC(FUN,X0,...) returns the value of the Hessian of the objective function FUN at the solution X.

Examples

FUN can be specified using @:

```
X = fminunc(@myfun,2)
```

where MYFUN is a MATLAB function such as:

```
function F = myfun(x)
F = sin(x) + 3;
```

To minimize this function with the gradient provided, modify the MYFUN so the gradient is the second output argument:

```
function [f,g]= myfun(x)
 f = sin(x) + 3;
 g = cos(x);
```

and indicate the gradient value is available by creating an options structure with OPTIONS.GradObj set to 'on' (using OPTIMSET):

```
options = optimset('GradObj','on');
x = fminunc('myfun',4,options);
```

FUN can also be an inline object:

```
x = fminunc(inline('sin(x)+3'),4);
```

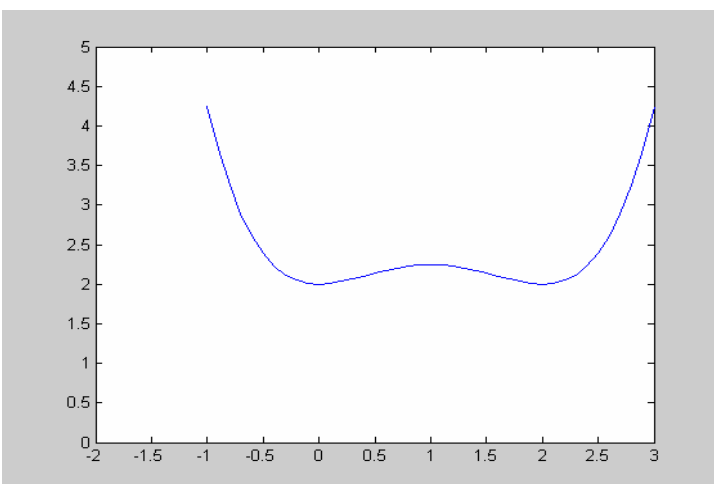
See also OPTIMSET, FMINSEARCH, FMINBND, FMINCON, @, INLINE.

Example 1

Minimize the function $y = \frac{1}{4}x^4 - x^3 + x^2 + 2.$

Note the derivative of y w.r.t. x is g= $x^3 - 3x^2 + 2x,$ which is equal to zero for -1, 0 and 2.

```
function xn=example1(x0)
x=-2:.1:3;
y=(1/4)*x.^4-x.^3+x.^2+2;
g=x.^3-3*x.^2+2*x;
[x;y;g]
plot(x,y)
axis([-2 3 0 5])
options=optimset('Diagnostics','on','Display','iter','GradObj','off','Hessian','off',…
        'LargeScale','off','DerivativeCheck','on','TolFun',1E-8);
[xn,fval,exitflag,output,grad] = fminunc('func1',x0,options)
if exitflag ~= 1
    xn,fval,exitflag,output,grad,error('convergence error')
end

function f=func1(x)
f=(1/4)*x^4-x^3+x^2+2;
```



```
>> xn=example1(-3)

ans =
  Columns 1 through 8

   -1.0000   -0.9000   -0.8000   -0.7000   -0.6000   -0.5000   -0.4000   -0.3000
    4.2500    3.7030    3.2544    2.8930    2.6084    2.3906    2.2304    2.1190
   -6.0000   -4.9590   -4.0320   -3.2130   -2.4960   -1.8750   -1.3440   -0.8970

  Columns 9 through 16

   -0.2000   -0.1000         0    0.1000    0.2000    0.3000    0.4000    0.5000
    2.0484    2.0110    2.0000    2.0090    2.0324    2.0650    2.1024    2.1406
   -0.5280   -0.2310         0    0.1710    0.2880    0.3570    0.3840    0.3750

  Columns 17 through 24

    0.6000    0.7000    0.8000    0.9000    1.0000    1.1000    1.2000    1.3000
    2.1764    2.2070    2.2304    2.2450    2.2500    2.2450    2.2304    2.2070
    0.3360    0.2730    0.1920    0.0990         0   -0.0990   -0.1920   -0.2730

  Columns 25 through 32

    1.4000    1.5000    1.6000    1.7000    1.8000    1.9000    2.0000    2.1000
    2.1764    2.1406    2.1024    2.0650    2.0324    2.0090    2.0000    2.0110
   -0.3360   -0.3750   -0.3840   -0.3570   -0.2880   -0.1710         0    0.2310

  Columns 33 through 41

    2.2000    2.3000    2.4000    2.5000    2.6000    2.7000    2.8000    2.9000    3.0000
    2.0484    2.1190    2.2304    2.3906    2.6084    2.8930    3.2544    3.7030    4.2500
    0.5280    0.8970    1.3440    1.8750    2.4960    3.2130    4.0320    4.9590    6.0000

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
   Diagnostic Information

Number of variables: 1

Functions
 Objective:                    (1/4)*x^4-x^3+x^2+2
 Gradient:                     finite-differencing
 Hessian:                      finite-differencing (or Quasi-Newton)

Algorithm selected
   medium-scale: Quasi-Newton line search

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 End diagnostic information
```

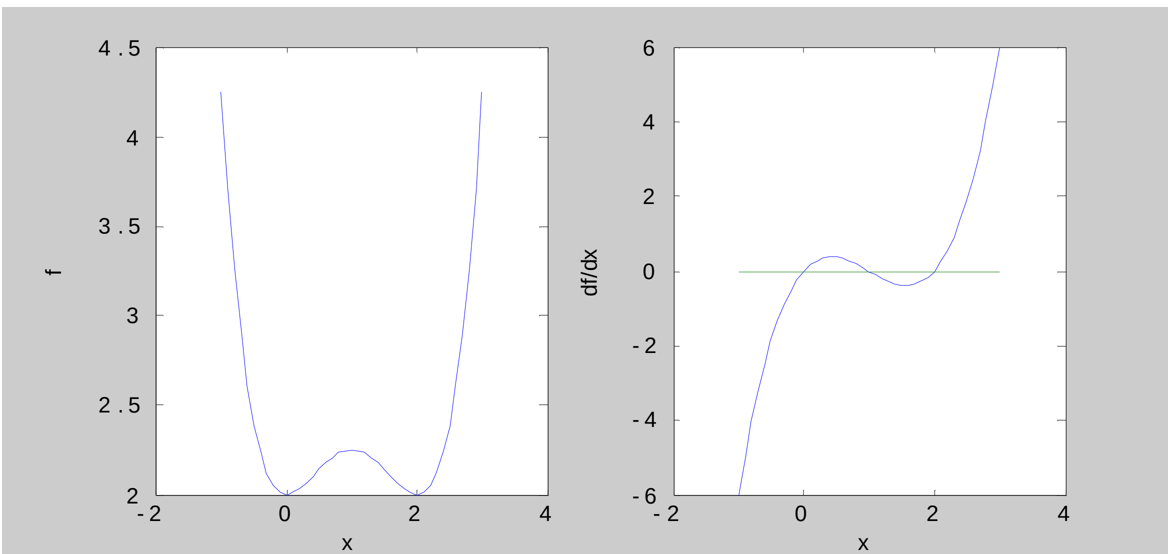
Iteration	Func-count	f(x)	Step-size	First-order optimality
0	2	58.25		60
1	4	18	0.0166667	24
2	6	6.93827	1	10.4
3	8	3.36258	1	4.26
4	10	2.34035	1	1.72
5	12	2.06774	1	0.642
6	14	2.00896	1	0.206
7	16	2.00054	1	0.0475
8	18	2.00001	1	0.00553
9	20	2	1	0.000184
10	22	2	1	7.15e-007
11	24	2	1	2.98e-008

```
Optimization terminated: relative infinity-norm of gradient less than options.TolFun.

xn =
    -2.4513e-009

fval =
    2

exitflag =
    1

output =
       iterations: 11
        funcCount: 24
         stepsize: 1
    firstorderopt: 2.9802e-008
        algorithm: 'medium-scale: Quasi-Newton line search'

grad =
    -2.9802e-008

ans=
    -2.4513e-009
```

short function:

```
function myfunc1(x,x0)
f=(1/4)*x.^4-x.^3+x.^2+2;
g=x.^3-3*x.^2+2*x;
subplot(1,2,1);plot(x,f)
ylabel('f');xlabel('x')
subplot(1,2,2);plot(x,g,x,(x-x))
ylabel('df/dx');xlabel('x')
xvalue=fminunc(inline('(1/4)*x.^4-x.^3+x.^2+2'),t)
```

```
>>x=-1:.1:3
>>myfunc1(x,-1)

xvalue=
     -1.1102e-016
```




```
>>myfunc(x,4)

xvalue=
          2.00000
```

Same plot

Remark: in the MATLAB output the x-axes are restricted to (-1,3).

Example 2

Minimize the function $f(\mathbf{x}) = (x_1^2 - x_2)^2 + (1 - x_1)^2$.

```
function xn=example2(x)
options=optimset('Diagnostics','on','Display','iter','GradObj','off',...
       'Hessian','off','LargeScale','off','DerivativeCheck','on','TolFun',1E-8);
[xn,fval,exitflag,output,grad] = fminunc('func2',x,options)
if exitflag ~= 1
   xn,fval,exitflag,output,grad,error('convergence error')
end

function [f,g]=func2(x)
f=(x(1)^2-x(2))^2+(1-x(1))^2;
```

```
>> xn=example2([.5 .5])

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Diagnostic Information

Number of variables: 2

Functions
 Objective:                    func2
 Gradient:                     finite-differencing
 Hessian:                      finite-differencing (or Quasi-Newton)

Algorithm selected
   medium-scale: Quasi-Newton line search

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 End diagnostic information

                                                      First-order
 Iteration  Func-count       f(x)        Step-size     optimality
     0           3          0.3125                           1.5
     1           9       0.0854609      0.177087           0.349
     2          12       0.0716593             1           0.238
     3          15       0.0331821             1           0.301
     4          18      0.00472458             1           0.175
     5          21     4.37497e-005            1          0.0246
     6          24     2.80563e-006            1         0.00424
     7          27     6.40314e-011            1        1.32e-005
     8          30     9.51925e-015            1        1.38e-007
     9          33      2.7754e-015            1        1.03e-010
Optimization terminated: relative infinity-norm of gradient less than options.TolFun.
```

```
xn =
    1.0000    1.0000

fval =
  2.7754e-015

exitflag =
     1

output =
          iterations: 9
        funcCount: 33
            stepsize: 1
    firstorderopt: 1.0337e-010
        algorithm: 'medium-scale: Quasi-Newton line search'
          message: [1x85 char]


grad =
  1.0e-009 *

   -0.1034
    0.0496

xn =
    1.0000    1.0000
```

Example 3: Use of gradients

```
function xn=example3(x)
options=optimset('Diagnostics','on','Display','iter','GradObj','on',...
      'Hessian','off','LargeScale','off','DerivativeCheck','on','TolFun',1E-8);
[xn,fval,exitflag,output,grad] = fminunc('func3',x,options)
if exitflag ~= 1
   xn,fval,exitflag,output,grad,error('convergence error')
end

function [f,g]=func3(x)
f=x(1)^2-x(1)*x(2)+x(2)^2;
g(1)=2*x(1)-x(2);
g(2)=-x(1)+2*x(2);


>> xn=example3([5 2])

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Diagnostic Information

Number of variables: 2

Functions
 Objective and gradient:             func3
 Hessian:                            finite-differencing (or Quasi-Newton)

Algorithm selected
   medium-scale: Quasi-Newton line search

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 End diagnostic information

Maximum discrepancy between derivatives  = 9.53674e-008
                                                        First-order
 Iteration  Func-count       f(x)        Step-size       optimality
     0           3               19                               8
     1           4          12.0156          0.125            5.88
     2           5          1.12689              1            1.72
     3           6         0.183274              1           0.829
     4           7       0.00113675              1          0.0637
     5           8     8.91309e-006              1         0.00585
     6           9     2.63756e-009              1      6.35e-005
     7          10     9.19743e-013              1      1.81e-006
     8          11     1.21433e-017              1      6.44e-009
Optimization terminated: relative infinity-norm of gradient less than options.TolFun.
```

```
xn =
  1.0e-008 *

    0.2455   -0.1534

fval =
  1.2143e-017

exitflag =
     1

output =
          iterations: 8
           funcCount: 11
            stepsize: 1
      firstorderopt: 6.4432e-009
           algorithm: 'medium-scale: Quasi-Newton line search'
             message: [1x85 char]

grad =
  1.0e-008 *

    0.6443
   -0.5522

xn =
  1.0e-008 *

    0.2455   -0.1534
```

Examples 4 and 5: Factor Analysis

Assume we have 4 variables and want to fit a one factor model. In the example here the sample size is 649. So the model equations are

$$\mathbf{\Sigma} = \boldsymbol{\lambda}\boldsymbol{\lambda}' + \mathbf{\Psi},$$

where $\mathbf{\Psi}$ is a diagonal matrix.
Furthermore, the least squares function can be written as

$$f = \sum_{i=1}^{p}\sum_{j=1}^{p}(s_{ij} - \sigma_{ij})^2 = tr(\mathbf{S} - \mathbf{\Sigma})^2$$

The vector “x” in MATLAB contains here first the 4 factor loadings and then the 4 error variances.

```
function example4
S=[86.3979 57.7751 56.8651 58.8986;….
     57.7751 86.2632 59.3177 59.6683;…
     56.8651 59.3177 97.2850 73.8201;…
     58.8986 59.6683 73.8201 97.8192]
x=rand(1,8)
options=optimset('Diagnostics','on','Display','iter','GradObj','off',...
       'Hessian','off','LargeScale','off','DerivativeCheck','on','TolFun',1E-8);
[xn,fval,exitflag,output,grad] = fminunc('func4',x,options,S)
if exitflag ~= 1
    xn,fval,exitflag,output,grad,error('convergence error')
end
'solution'
LAB=xn(1:4)'
PSI=diag(xn(5:8))
fval

function f=func4(x,S)
LAB=x(1:4)';
PSI=diag(x(5:8));
SIG=LAB*LAB'+PSI;
f=trace((S-SIG)^2);
```

```
>> example4

S =
   86.3979   57.7751   56.8651   58.8986
   57.7751   86.2632   59.3177   59.6683
   56.8651   59.3177   97.2850   73.8201
   58.8986   59.6683   73.8201   97.8192

x =
    0.8214    0.4447    0.6154    0.7919    0.9218    0.7382    0.1763    0.4057

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
   Diagnostic Information

Number of variables: 8

Functions
 Objective:                          func4
 Gradient:                          finite-differencing
 Hessian:                           finite-differencing (or Quasi-Newton)


Algorithm selected
   medium-scale: Quasi-Newton line search

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 End diagnostic information

                                                        First-order
 Iteration  Func-count       f(x)        Step-size       optimality
     0           9         77691.1                          784
     1          36         2900.05      0.0100889           781
     2          63         2597.36      0.0134495           350
     3          81         2500.67       0.123056          49.2
     4          99          1729.6             10           266
     5         117         973.247         0.1484           758
     6         126         252.255              1           472
     7         144          158.88            0.1           125
     8         153         116.702              1            26
     9         162         116.259              1           3.5
    10         171         116.242              1         0.636
    11         180         116.229              1          1.78
    12         189         116.173              1          6.07
    13         198         116.092              1             9
    14         207         115.989              1          8.47
    15         216         115.937              1          3.98
    16         225         115.927              1         0.677
    17         234         115.926              1        0.0216
```

```
                                                First-order
 Iteration  Func-count       f(x)        Step-size       optimality

    18         243         115.926                1         0.0206
    19         252         115.926                1         0.0272
    20         261         115.926                1         0.0416
    21         270         115.926                1         0.0508
    22         279         115.926                1         0.0434
    23         288         115.926                1         0.0195
    24         297         115.926                1        0.00316
    25         306         115.926                1        0.00285
    26         315         115.926                1        0.00271
    27         324         115.926                1        0.00255
    28         333         115.926                1        0.00251
    29         342         115.926                1        0.00195
    30         351         115.926                1       0.000894
    31         360         115.926                1       0.000338
    32         369         115.926                1      5.84e-005
    33         378         115.926                1      5.16e-006
Optimization terminated: relative infinity-norm of gradient less than options.TolFun.

xn =
    7.1752    7.3601    8.2932    8.4502   34.9144   32.0915   28.5079   26.4140

fval =
  115.9256

exitflag =
     1

output =
       iterations: 33
        funcCount: 378
         stepsize: 1
    firstorderopt: 5.1630e-006
        algorithm: 'medium-scale: Quasi-Newton line search'

grad =
  1.0e-005 *

   -0.0665
   -0.0648
   -0.2415
    0.0113
    0.4698
   -0.1189
    0.0167
   -0.5163
```

```
ans =
solution

LAB =
    7.1752
    7.3601
    8.2932
    8.4502

PSI =
   34.9144         0         0         0
         0   32.0915         0         0
         0         0   28.5079         0
         0         0         0   26.4140

fval =
  115.9256
```

We now perform a 2 factor model of the form

$$\mathbf{\Sigma} = \mathbf{\Lambda\Phi\Lambda}' + \mathbf{\Psi},$$

where the matrices have the following form

$$\mathbf{\Lambda} = \begin{pmatrix} \alpha & 0 \\ \alpha & 0 \\ 0 & \beta \\ 0 & \beta \end{pmatrix}$$
$$\mathbf{\Phi} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$
$$\mathbf{\Psi} = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}$$

In the MATLAB program we define the vector x as: $x = (\alpha \quad \beta \quad \gamma \quad \delta \quad \rho)$. So there are 5 unknown parameters. In the program we start with random start values.

```
function example5
S=[86.3979 57.7751 56.8651 58.8986;...
     57.7751 86.2632 59.3177 59.6683;...
     56.8651 59.3177 97.2850 73.8201;...
     58.8986 59.6683 73.8201 97.8192]
x=rand(1,5)
options=optimset('Diagnostics','on','Display','iter','GradObj','off',...
      'Hessian','off','LargeScale','off','DerivativeCheck','on','TolFun',1E-8);
[xn,fval,exitflag,output,grad] = fminunc('func5',x,options,S)
if exitflag ~= 1
   xn,fval,exitflag,output,grad,error('convergence error')
end
'solution'
LAB=[xn(1) 0;xn(1) 0;0 xn(2);0 xn(2)]
PHI=[1 xn(5);xn(5) 1]
PSI=diag([xn(3) xn(3) xn(4) xn(4)])
SIG=LAB*PHI*LAB'+PSI
S-SIG
f=func5(xn,S)
```

```
function f=func5(x,S)
LAB=[x(1) 0;x(1) 0;0 x(2);0 x(2)];
PHI=[1 x(5);x(5) 1];
PSI=diag([x(3) x(3) x(4) x(4)]);
SIG=LAB*PHI*LAB'+PSI;
f=trace((S-SIG)^2);


>> example5

S =
   86.3979   57.7751   56.8651   58.8986
   57.7751   86.2632   59.3177   59.6683
   56.8651   59.3177   97.2850   73.8201
   58.8986   59.6683   73.8201   97.8192

x =
    0.8216    0.6449    0.8180    0.6602    0.3420

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
   Diagnostic Information

Number of variables: 5

Functions
 Objective:                        func5
 Gradient:                         finite-differencing
 Hessian:                          finite-differencing (or Quasi-Newton)

Algorithm selected
   medium-scale: Quasi-Newton line search

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
 End diagnostic information

                                                        First-order
 Iteration  Func-count       f(x)        Step-size       optimality
     0           6         77700.6                       1.14e+003
     1          36         21337.2      0.00416643       6.56e+003
     2          54         21082.8       0.0431329       9.69e+003
     3          66         5824.02             0.5       3.87e+003
     4          84         4887.96        0.091649       1.13e+004
     5          90         2429.54               1       7.31e+003
     6          96         1178.28               1       3.02e+003
     7         102         1069.89               1       1.24e+003
     8         108         1055.38               1             143
     9         114         1050.72               1            97.2
    10         120         1048.97               1              49
```

```
                                                        First-order
 Iteration  Func-count       f(x)        Step-size       optimality
    11           126        1041.28            1            164
    12           132        1024.36            1            420
    13          138         978.007            1            842
    …            …          ………              .             …
    31           246        9.60156            1           1.74
    32           252        9.60141            1         0.0904
    33           258        9.60141            1         0.0188
    34           264        9.60141            1       0.000627
    35           270        9.60141            1      2.38e-006
Optimization terminated: relative infinity-norm of gradient less than options.TolFun.

xn =
    7.6010    8.5919   28.5555   23.7320    0.8986

fval =
    9.6014

exitflag =
     1

output =
          iterations: 35
           funcCount: 270
            stepsize: 1
       firstorderopt: 2.3842e-006
           algorithm: 'medium-scale: Quasi-Newton line search'

grad =
  1.0e-005 *

   -0.0235
   -0.1804
   -0.0083
   -0.0045
   -0.2384

ans =
solution

LAB =
    7.6010         0
    7.6010         0
         0    8.5919
         0    8.5919
```

```
PHI =
    1.0000    0.8986
    0.8986    1.0000

PSI =
   28.5555         0         0         0
         0   28.5555         0         0
         0         0   23.7320         0
         0         0         0   23.7320

SIG =
   86.3305   57.7751   58.6874   58.6874
   57.7751   86.3305   58.6874   58.6874
   58.6874   58.6874   97.5521   73.8201
   58.6874   58.6874   73.8201   97.5521

ans =
    0.0674    0.0000   -1.8223    0.2112
    0.0000   -0.0673    0.6303    0.9809
   -1.8223    0.6303   -0.2671    0.0000
    0.2112    0.9809    0.0000    0.2671

f =
    9.6014
```

Example 6: Maximum likelihood estimates

The function to be minimized which yields maximum likelihood estimates is

$$f = tr(\mathbf{\Sigma}^{-1}\mathbf{S}) + \log\left(\left|\mathbf{\Sigma}^{-1}\mathbf{S}\right|\right) - p,$$

where p is the number of variables; here $p = 4$. A problem with this function may be that during the iterative process the determinant may become negative, i.e. when some of the eigenvalues become negative. Therefore a good start vector is needed. In this example we first compute the least squares estimates of the parameters, and then use this output as starting vector for the maximum likelihood method.

P.S. Under the assumption of normality of the variables it holds that the final function value*(n-1) is chi-square distributed with degrees of freedom equal to the total number of (co)variances in the observed covariance matrix (i.e. p(p+1)/2) minus the number of unknown parameters. So in this case df=10 – 5 = 5.

```
function example6
S=[86.3979 57.7751 56.8651 58.8986;...
    57.7751 86.2632 59.3177 59.6683;...
    56.8651 59.3177 97.2850 73.8201;...
    58.8986 59.6683 73.8201 97.8192]
ind=0
x=rand(1,5)
options=optimset('Diagnostics','off','Display','off','GradObj','off',...
     'Hessian','off','LargeScale','off','DerivativeCheck','on','TolFun',1E-6);
[xn,fval,exitflag,output,grad] = fminunc('func6',x,options,S,ind)
if exitflag ~= 1
   xn,fval,exitflag,output,grad, error('convergence error')
end
ind=1
[xn,fval,exitflag,output,grad] = fminunc('func6',xn,options,S,ind)
if exitflag ~= 1
   xn,fval,exitflag,output,grad, error('convergence error')
end
'solution'
LAB=[xn(1) 0;xn(1) 0;0 xn(2);0 xn(2)]
PHI=[1 xn(5);xn(5) 1]
PSI=diag([xn(3) xn(3) xn(4) xn(4)])
SIG=LAB*PHI*LAB'+PSI
S-SIG
X2=648*func6(xn,S,ind); %  648=n-1
df=10-length(x);
pvalue=1-chi2cdf(X2,df);
‘Chi-square, df, p-value’
[X2 df pvalue]
```

```
function f=func6(x,S,ind)
LAB=[x(1) 0;x(1) 0;0 x(2);0 x(2)];
PHI=[1 x(5);x(5) 1];
PSI=diag([x(3) x(3) x(4) x(4)]);
SIG=LAB*PHI*LAB'+PSI;
if ind == 0
   f=trace((S-SIG)^2);
else
   A=inv(SIG)*S;
   f=trace(A)-log(det(A))-4;
end
```

```
>> example6

S =
   86.3979   57.7751   56.8651   58.8986
   57.7751   86.2632   59.3177   59.6683
   56.8651   59.3177   97.2850   73.8201
   58.8986   59.6683   73.8201   97.8192

ind =
     0

x =
    0.1536    0.6756    0.6992    0.7275    0.4784

xn =
    7.6010    8.5919   28.5555   23.7320    0.8986

fval =
    9.6014

exitflag =
     1


output =

       iterations: 32
        funcCount: 246
         stepsize: 1
    firstorderopt: 9.4175e-006
        algorithm: 'medium-scale: Quasi-Newton line search'
```

```
grad =
  1.0e-005 *

   -0.1364
    0.0583
   -0.0058
    0.0146
    0.9418

ind =
     1

xn =
    7.6010    8.5919   28.5555   23.7320    0.8986

fval =
    0.0030

exitflag =
     1

output =
       iterations: 0
        funcCount: 6
         stepsize: []
    firstorderopt: 1.7881e-007
        algorithm: 'medium-scale: Quasi-Newton line search'
          message: [1x117 char]

grad =
  1.0e-006 *

         0
         0
    0.0021
    0.0025
    0.1788

ans =
solution

LAB =
    7.6010         0
    7.6010         0
         0    8.5919
         0    8.5919
```

```
PHI =
    1.0000    0.8986
    0.8986    1.0000

PSI =
   28.5555         0         0         0
         0   28.5555         0         0
         0         0   23.7320         0
         0         0         0   23.7320

SIG =
   86.3305   57.7751   58.6874   58.6874
   57.7751   86.3305   58.6874   58.6874
   58.6874   58.6874   97.5521   73.8201
   58.6874   58.6874   73.8201   97.5521

ans =
    0.0674    0.0000   -1.8223    0.2112
    0.0000   -0.0673    0.6303    0.9809
   -1.8223    0.6303   -0.2671    0.0000
    0.2112    0.9809    0.0000    0.2671

ans =
Chi-square, df, p-value

ans =
    1.9335    5.0000    0.8583
```

Thus: the chi square value is 1.9335, the degrees of freedom is 5 and the p-level is .8583
.

Example 7: Extra, minimize function with all kinds of constraints

Minimize

$$f = (\mathbf{x}-\mathbf{m})'(\mathbf{x}-\mathbf{m}) \text{ with } \mathbf{m} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 3 \\ 1 \end{pmatrix} \text{ and } \mathbf{Ax} \le \mathbf{0}, \text{ where}$$

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

The inequality constraints mean that $x_1 \le x_2 \le x_3 \le x_4 \le x_5.$ Verify that. The inequality constraints mean that the vector **x** is in increasing order. Without these inequality constraints the optimum is **x** = **m**.

For finding the solution with inequality constraints we use the MATLAB function "fmincon". By this function all kinds of constraints can be imposed on the vector **x.** Like lower bounds, upper bounds, equality constraints, inequality constraints, nonlinear equalities and nonlinear inequalities. We don't discuss these possibilities here. Only the problem above with inequality constraints will be shown.

```
>> help fmincon
 FMINCON finds a constrained minimum of a function of several variables.
   FMINCON attempts to solve problems of the form:
       min F(X)  subject to:  A*X  <= B, Aeq*X  = Beq (linear constraints)
        X                    `      C(X) <= 0, Ceq(X) = 0   (nonlinear constraints)
                                    LB <= X <= UB

   X=FMINCON(FUN,X0,A,B) starts at X0 and finds a minimum X to the function
   FUN, subject to the linear inequalities A*X <= B. FUN accepts input X and
   returns a scalar function value F evaluated at X. X0 may be a scalar,
   vector, or matrix.

   X=FMINCON(FUN,X0,A,B,Aeq,Beq) minimizes FUN subject to the linear equalities
   Aeq*X = Beq as well as A*X <= B. (Set A=[] and B=[] if no inequalities exist.)

   X=FMINCON(FUN,X0,A,B,Aeq,Beq,LB,UB) defines a set of lower and upper
   bounds on the design variables, X, so that a solution is found in
   the range LB <= X <= UB. Use empty matrices for LB and UB
   if no bounds exist. Set LB(i) = -Inf if X(i) is unbounded below;
   set UB(i) = Inf if X(i) is unbounded above.

   X=FMINCON(FUN,X0,A,B,Aeq,Beq,LB,UB,NONLCON) subjects the minimization to the
   constraints defined in NONLCON. The function NONLCON accepts X and returns
   the vectors C and Ceq, representing the nonlinear inequalities and equalities
   respectively. FMINCON minimizes FUN such that C(X)<=0 and Ceq(X)=0.
   (Set LB=[] and/or UB=[] if no bounds exist.)

  X=FMINCON(FUN,X0,A,B,Aeq,Beq,LB,UB,NONLCON,OPTIONS) minimizes with the
  default optimization parameters replaced by values in the structure OPTIONS, an argument
  created with the OPTIMSET function. See OPTIMSET for details. Used options are Display,
  TolX, TolFun, TolCon, DerivativeCheck, Diagnostics, FunValCheck, GradObj, GradConstr,
  Hessian, MaxFunEvals, MaxIter, DiffMinChange and DiffMaxChange, LargeScale,
  MaxPCGIter, PrecondBandWidth, TolPCG, TypicalX, Hessian, HessMult, HessPattern, and
  OutputFcn. Use the GradObj option to specify that FUN also returns a second output argument
  G that is the partial derivatives of the function df/dX, at the point X. Use the Hessian option to
  specify that FUN also returns a third output argument H that is the 2nd partial derivatives of the
  function (the Hessian) at the point X. The Hessian is only used by the large-scale method, not
  the line-search method. Use the GradConstr option to specify that NONLCON also returns third
  and fourth output arguments GC and GCeq, where GC is the partial derivatives of the
  constraint vector of inequalities C, and GCeq is the partial derivatives of the constraint vector of
  equalities Ceq. Use OPTIONS = [] as a  place holder if no options are set.

   [X,FVAL]=FMINCON(FUN,X0,...) returns the value of the objective function FUN at the solution X.

   [X,FVAL,EXITFLAG]=FMINCON(FUN,X0,...) returns an EXITFLAG that describes the
   exit condition of FMINCON. Possible values of EXITFLAG and the corresponding exit conditions are

     1  First order optimality conditions satisfied to the specified tolerance.
     2  Change in X less than the specified tolerance.
     3  Change in the objective function value less than the specified tolerance.
     4  Magnitude of search direction smaller than the specified tolerance and
        constraint violation less than options.TolCon.
     5  Magnitude of directional derivative less than the specified tolerance
        and constraint violation less than options.TolCon.
     0  Maximum number of function evaluations or iterations reached.
    -1  Optimization terminated by the output function.
    -2  No feasible point found.
```

```
[X,FVAL,EXITFLAG,OUTPUT]=FMINCON(FUN,X0,...) returns a structure
OUTPUT with the number of iterations taken in OUTPUT.iterations, the number
of function evaluations in OUTPUT.funcCount, the algorithm used in
OUTPUT.algorithm, the number of CG iterations (if used) in OUTPUT.cgiterations,
the first-order optimality (if used) in OUTPUT.firstorderopt, and the exit
message in OUTPUT.message.

[X,FVAL,EXITFLAG,OUTPUT,LAMBDA]=FMINCON(FUN,X0,...) returns the Lagrange multipliers
at the solution X: LAMBDA.lower for LB, LAMBDA.upper for UB, LAMBDA.ineqlin is
for the linear inequalities, LAMBDA.eqlin is for the linear equalities, LAMBDA.ineqnonlin is for
the nonlinear inequalities, and LAMBDA.eqnonlin is for the nonlinear equalities.

[X,FVAL,EXITFLAG,OUTPUT,LAMBDA,GRAD]=FMINCON(FUN,X0,...) returns the value of
the gradient of FUN at the solution X.

[X,FVAL,EXITFLAG,OUTPUT,LAMBDA,GRAD,HESSIAN]=FMINCON(FUN,X0,...) returns the
value of the HESSIAN of FUN at the solution X.

Examples
  FUN can be specified using @:
     X = fmincon(@humps,...)
  In this case, F = humps(X) returns the scalar function value F of the HUMPS function
  evaluated at X.

  FUN can also be an anonymous function:
     X = fmincon(@(x) 3*sin(x(1))+exp(x(2)),[1;1],[],[],[],[],[0 0])
  returns X = [0;0].

If FUN or NONLCON are parameterized, you can use anonymous functions to capture
the problem-dependent parameters. Suppose you want to minimize the objective
given in the function myfun, subject to the nonlinear constraint mycon, where
these two functions are parameterized by their second argument a1 and a2, respectively.
Here myfun and mycon are M-file functions such as

     function f = myfun(x,a1)
     f = x(1)^2 + a1*x(2)^2;

and

     function [c,ceq] = mycon(x,a2)
     c = a2/x(1) - x(2);
     ceq = [];

To optimize for specific values of a1 and a2, first assign the values to these
two parameters. Then create two one-argument anonymous functions that capture
the values of a1 and a2, and call myfun and mycon with two arguments. Finally,
pass these anonymous functions to FMINCON:

     a1 = 2; a2 = 1.5; % define parameters first
     options = optimset('LargeScale','off'); % run medium-scale algorithm
     x = fmincon(@(x)myfun(x,a1),[1;2],[],[],[],[],[],[],@(x)mycon(x,a2),options)

See also optimset, fminunc, fminbnd, fminsearch, @, function_handle.

Reference page in Help browser
   doc fmincon
```

```
function example7
lb=[],ub=[]
'no lower or upper bound'
A=[1 -1 0 0 0;...
       0 1 -1 0 0;...
       0 0 1 -1 0;...
       0 0 0 1 -1]
b=[0;0;0;0]
'Ax<=b inequality constraints'
Aeq=[], beq=[]
'no linear equality constraints'
options=optimset('Diagnostics','off','Display','off','GradObj','off',...
   'Hessian','off','LargeScale','off','DerivativeCheck','on','TolFun',1E-6);
x=rand(1,5)
 [xn,fval,exitflag,output,grad] = ...
    fmincon(@func7,x,A,b,Aeq,beq,lb,ub,@funcon7a,options)
'funcon7a no nonlinear (in)equality constraints'
'funcon7b only nonlinear inequality constraints'
if exitflag ~= 1
    xn,fval,exitflag,output,grad,error('convergence error')
end

function f=func7(x)
m=[1 3 2 3 1];
f=sum((x-m).^2);

function [c,ceq]=funcon7a(x)
c=[];
% no nonlinear inequality constraints
ceq=[];
% no nonlinear equality constraints

function [c,ceq]=funcon7b(x)
c=[x(1)-x(2);...
   x(2)-x(3)];
% nonlinear inequality constraints
ceq=[];
% no nonlinear equality constraints
```

```
>> example7
lb =
     []
ub =
     []
ans =
no lower or upper bound
```

```
A =
     1    -1     0     0     0
     0     1    -1     0     0
     0     0     1    -1     0
     0     0     0     1    -1
b =
     0
     0
     0
     0
ans =
Ax<=b inequality constraints
Aeq =
     []
beq =
     []

ans =
funcon7a no nonlinear (in)equality constraints
ans =
funcon7b only nonlinear inequality constraints

x =
    0.0579    0.3529    0.8132    0.0099    0.1389

xn =
    1.0000    2.2500    2.2500    2.2500    2.2500
fval =
    2.7500
exitflag =
     1
output =
         iterations: 3
          funcCount: 24
           stepsize: 1
          algorithm: 'medium-scale: SQP, Quasi-Newton, line-search'
      firstorderopt: 3.3939e-008
       cgiterations: []
            message: [1x144 char]

grad =
          lower: [5x1 double]
          upper: [5x1 double]
          eqlin: [0x1 double]
       eqnonlin: [0x1 double]
         ineqlin: [4x1 double]
     ineqnonlin: [0x1 double]
```

Exercises chapter 4

1: Minimize the function $f(x) = 2x^2 - e^x$. Note: the derivative is $g(x) = 4x - e^x$. Solving the zero points of this gradient is not simple. Therefore for minimizing the function $f(x) = 2x^2 - e^x$ we use an iterative algorithm; as MATLAB does.

2: Minimize the function $f(\mathbf{x}) = (x_2 - x_1)^4 + 8x_1x_2 - x_1 + x_2 + 3.$ Make a plot of this function and show that there are a local minimum, a global minimum and a saddle point.

3. Minimize the function $f(\mathbf{x}) = x_1^2 - x_1x_2 + x_2^2.$ For a plot of this function see example 5 of chapter 3. Compare the results.

4: Minimize the function $f(\mathbf{x}) = .5x_1^2 - 2x_1x_2 + 2x_2^2.$ For a plot of this function see example 5 of chapter 3. Compare the results.

5: Minimize the function $f(\mathbf{x}) = x_1^2 - 2x_1x_2 + \frac{1}{2}(x_2^2 - 1).$ For a plot of this function see example 5 of chapter 3. Compare the results.

6: Instead of carrying out an unweighted least squares method in example 4 above, carry out a maximum likelihood estimation method. Does the model fit the data?

7: In example 6 above the model did fit the data. The corresponding model has a correlation between the factors named ρ which was estimated as $\hat{\rho} = .8986.$ Fit a model in which $\rho = 1.$ Does this model fit the data?