Matrix Algebra 8 weeks - test
Matlab 7 weeks – assignments

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Test: Monday, 6-11-2006, 11-13 hours
Retest: Monday, 18-12-2006, 11-13 hours

Matrix Algebra

1. Some Basics - matrix
2. The determinant
3. The inverse
4. Set of equations
5. Eigenvectors and eigenvalues
6. Applications in Statistics

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Organisatie
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Onderwijs
Mastercursussen
Matrix Algebra & Matlab

- **An Introduction to Matrix Algebra with Matlab**

1. Some basics

1.1 Matrix: table with numbers

A of order (2 × 3) or (8 × 8) or (p × n)

Rectangular matrix (2 × 3)

Square matrix (8 × 8)

- Diagonal matrix
- Identity matrix
- symmetric or asymmetric matrix

Null matrix

row vectors of **A**, number: 2, 8, p

column vectors of **A**, number: 3, 8, n

A (2 × 3) element $\mathbf{a}_{ij}$ or **a**(i,j)

A′ transpose of **A** (3 × 2)

x column vector (n×1) or (n)

x′ row vector (1×n)

1.2 Simple operations

A (2 × 3) and **B** (3 × 2)
<u>**A** and **B**′ same orders:</u>
A+**B**′ add element wise
A-**B**′ subtract element wise
A+B not possible

Scalar product: c
c**A** each element times c

vector **a**′ (1×3) first row of **A**(2×3)
vector **b** (3×1) or (3) first column of **B**(3×2)

a′b=scalar (1×3)(3×1) →(1×1)
Ab (2×3)(3×1) →(2×1) or (2) column vector
a′B (1×3)(3×2) →(1×2) row vector

AB (2×3)(3×2) →(2×2)

$\mathbf{a}'(1\times p)$, $\mathbf{b}(p\times 1)$

$$\mathbf{a}'\mathbf{b} = \sum_{i=1}^{p} a'_{1i} \times b_{i1}$$

$\mathbf{a}'$ $(1\times p)$, $\mathbf{B}(p\times k)$

$$\mathbf{a}'\mathbf{b}_{r} = \sum_{i=1}^{p} a'_{1i} \times b_{ir}$$

r=1,..,k

$\mathbf{a}'\mathbf{B}(1\times k)$

$\mathbf{A}(n\times p)$ and $\mathbf{B}(p\times k)$

$$\mathbf{AB}_{jr} = \sum_{i=1}^{p} a_{ji} \times b_{ir}$$

j=1,..,n and r=1,..,k

$\mathbf{AB}(n\times k)$

D diagonal (3×3)
A (2×3) and **B** (3×2)

AD each column j multiplied by d(j,j)
DB each row i multiplied by d(i,i)

If **D=I**
AI=A and **IB=B**

*In general***:**
A+B=B+A
A-B=-1**(B-A)**
AB≠BA

1.3 Examples

def **B** p.8
Compute **B′**

Def **A,B,y,c** top p.9
Compute **A+B, A-B, AB,** c**A,** order of **A**

Exercises

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & 9 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 4 & 3 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

1. $\mathbf{A}+\mathbf{B}$
2. $\mathbf{B}'+\mathbf{C}$
3. $\mathbf{B}-\mathbf{C}$
4. $\mathbf{x}'\mathbf{A}$
5. $(\mathbf{A}\mathbf{y})'$
6. $\mathbf{B}\mathbf{y}$
7. $\mathbf{B}'\mathbf{B}$
8. $3\mathbf{C}$
9. $\mathbf{A}\mathbf{B}$

Answers:

1. **A+B ??? Error**
2. $\mathbf{B'+C} = \begin{pmatrix} 2 & 4 & 5 \\ 2 & 7 & 6 \end{pmatrix}$
3. **B-C ??? Error**
4. **x'A = (7 17 29)**
5. **(Ay)’ = (8 11 19)**
6. **By ??? Error**
7. $\mathbf{B'B} = \begin{pmatrix} 21 & 20 \\ 20 & 25 \end{pmatrix}$
8. $\mathbf{3C} = \begin{pmatrix} 3 & 6 & 3 \\ 6 & 9 & 9 \end{pmatrix}$
9. $\mathbf{AB} = \begin{pmatrix} 17 & 17 \\ 27 & 27 \\ 48 & 47 \end{pmatrix}$

Home:
1.4 Exercises
Bring the answers

Chapter 2 Determinant

1.2 Definition

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad square(2 \times 2)$$

determinant **A=|A|=**$a_{11}a_{22}$-$a_{12}a_{21}$

A square (p×p)
minor of a_{ij}, determinant of **A** if row i and column j deleted
cofactor $c_{ij}=(-1)^{i+j} \times$ minor of a_{ij}

Example

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix}$$

Minor of a_{11}, a_{22}, a_{23}

$$|\mathbf{A}_{11}| = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = ... \; |\mathbf{A}_{22}| = ... \; |\mathbf{A}_{23}| = ...$$

Cofactors of a_{11}, a_{22}, a_{23}

$$\mathbf{c}_{11} = (-1)^{1+1}\left|\mathbf{A}_{\mathbf{22}}\right| = (-1)^{2}\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = \ldots\ldots\ldots$$

$$\mathbf{c}_{22} = (-1)^{2+2}\left|\mathbf{A}_{\mathbf{22}}\right| = \ldots\ldots\ldots$$

$$\mathbf{c}_{\mathbf{23}} = (-1)^{2+3}\left|\mathbf{A}_{\mathbf{23}}\right| = \ldots\ldots\ldots$$

Determinant of **A**

$$|\mathbf{A}| = \sum_{j=1}^{p} a_{ij} \times c_{ij} \quad in\ terms\ of\ row\ i$$

$$|\mathbf{A}| = \sum_{i=1}^{p} a_{ij} \times c_{ij} \quad in\ terms\ of\ column\ j$$

Exercise 1, p.12

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix}$$

Det (**A**)=-15 (using row 1)

el	det	cof	el×cof
$a_{11}=1$	$\lvert A_{11}\rvert$=..	+( )	1×…= …
$a_{12}=2$	$\lvert A_{12}\rvert$=..	-( )	2×…=…
$a_{13}=3$	$\lvert A_{13}\rvert$=..	+( )	3×…=<u>…</u>
			sum -15 p.12

- Compute det(**A**) by using **column 3**
- Compute det(**A**) by using **row 2**

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \\ 3 & 1 & 4 \end{pmatrix}$$

Column 3

el	det	cof	el×cof
$a_{13}=3$	$\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = -4$	-4	3×(-4)=-12
$a_{23}=1$	$\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$	+5	1×(+5)=+5
$a_{33}=4$	$\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = -2$	-2	4×(-2)=-8
		sum	-15

Row 2

el	det	cof	el×cof
$a_{21}=2$	$\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 5$	-5	2×(-5)=-10
$a_{22}=2$	$\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = -5$	-5	2×(-5)=-10
$a_{23}=1$	$\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$	+5	1×(+5)=+5
		sum	-15

2.4 Properties

1 $|\mathbf{A}|=|\mathbf{A}'|$

2 One row/column 0→ $|\mathbf{A}|=0$

3 $|\mathbf{AB}|=|\mathbf{A}||\mathbf{B}|$

4 **A** upper or under triangular

$$|\mathbf{A}| = \prod_{i=1}^{p} a_{ii}$$

5 **B** formed by interchanging two rows/columns of **A**→$|\mathbf{B}|=-|\mathbf{A}|$

6 **B** formed by one row of **A** ×**k** →$|\mathbf{B}|=\mathbf{k}|\mathbf{A}|$

7 **B** formed by one r/c of **A** ×**k** and adding this to another r/c →$|\mathbf{B}|=|\mathbf{A}|$

8 If two rows/columns of **A** are equal→$|\mathbf{A}|=0$

Prop1

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \qquad |\mathbf{A}| = 2(1) + 4(3) = 14$$

$$\mathbf{A}' = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 4 & 1 & 3 \end{pmatrix} \qquad |\mathbf{A}'| = 2(1) + 4(3) = 14$$

Prop3

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \qquad |\mathbf{A}| = -4$$

$$\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \qquad |\mathbf{B}| = 1$$

AB

$$\mathbf{AB} = \begin{pmatrix} 5 & 7 \\ 7 & 9 \end{pmatrix}$$

$$|\mathbf{AB}| = 45 - 49 = -4$$

Prop7

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 3 \\ 4 & 5 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

First row B: row1+2×row2

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 2 & 3 \\ 4 & 5 & 0 \\ 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1+8 & 2+10 & 3+0 \\ 4 & 5 & 0 \\ 2 & 0 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 9 & 12 & 3 \\ 4 & 5 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

Using column 3:

|B|=3(-10)+3(45-48)=-39

$$|\mathbf{B}| = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} 0 & 2 & 3 \\ 4 & 5 & 0 \\ 2 & 0 & 3 \end{vmatrix} = 1 \times \{3(-10) + 3(-3)\} = -39$$

Exercise 2

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 0 & 2 & 5 \\ 4 & 2 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

Compute det(**A**) by using column 1 (why?)

el	det	cof	el×cof
$a_{11}=1$	$\lvert A_{11}\rvert=\ldots$	+(...)	1×.. =……..
$a_{21}=3$	$\lvert A_{21}\rvert=\ldots$	-(…)	3×.. =……..
$a_{31}=4$	$\lvert A_{31}\rvert=\ldots$	+(...)	4×.. =……..
			sum …….

Answer exercise 2

$$\mathbf{A}_{11} = \begin{pmatrix} 0 & 2 & 5 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

el	det	cof	el×cof
2	$\begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix} = -4$	+4	8
1	$\begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = -3$	-3	<u>-3</u>
			5

$$\mathbf{A}_{21} = \begin{pmatrix} 2 & 0 & 4 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

el	det	cof	el×cof
2	$\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1$	+1	2
4	$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$	+3	<u>12</u>
			14

$$\mathbf{A}_{31} = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix}$$

el	det	cof	el×cof
2	$\begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix} = -4$	-4	-8
4	$\begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} = -2$	-2	-8
			-16

el	det	cof	el×cof	
$a_{11}=1$	$\lvert A_{11}\rvert=5$	+5	1×5	= 5
$a_{21}=3$	$\lvert A_{21}\rvert=14$	-14	3×(-14)	=-42
$a_{31}=4$	$\lvert A_{31}\rvert=-16$	-16	4×(-16)	=-64
			sum	-101

At home:
2.3 Exercises
Bring the answers

Chapter 3 Inverse

A square matrix
Inv(A)=$\mathbf{A}^{-1}$ than $\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}$ and $\mathbf{A}\mathbf{A}^{-1}=\mathbf{I}$

3.1 Linear equations

Ax=b, solve **x**, than $\mathbf{A}^{-1}\mathbf{A}\mathbf{x}=\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$

3.2 Alternative

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}(\mathbf{A}^{c})'$$

where $\mathbf{A}^c$ = matrix with **cofactors**
cofactor $c_{ij}=(-1)^{i+j} \times$ minor(i,j)
minor(i,j), det of A if row_i, column_j deleted

3.4 Properties of the inverse

1 If $|\mathbf{A}|\neq 0$ than $\mathbf{A}^{-1}$ unique
2 $(\mathbf{A}^{-1})^{-1}=\mathbf{A}$
3 $(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}$
4 **A** nonsingular, i.e. $|\mathbf{A}|\neq 0$, than $\mathbf{A}^{-1}$ nonsingular
5 $(\mathbf{A}')^{-1}=(\mathbf{A}^{-1})'$

Chapter 4
Linear (in)dependence of vectors

4.1 linear (in)dependence

vectors $\mathbf{a_1,..,a_n}$

scalars $k_1,..,k_n$

A set of vectors is **linear dependent** if $k_1\mathbf{a_1},.., k_n\mathbf{a_n} = \mathbf{0}$, and <u>not</u> all $k_i=0$

A set of vectors is **linear <u>in</u>dependent** if $k_1\mathbf{a_1},.., k_n\mathbf{a_n} = \mathbf{0}$, **and** all $k_i=0$

Ex1$\mathbf{Ak}=\mathbf{0}$ $k_1=0$ and $k_2=0$ →independent

Ex2 $\mathbf{Ak}=\mathbf{0}$ $k_1=-3k_2$, $0k_2=0$, many solutions → dependent

Ex3 if $\mathbf{a_i}=\mathbf{0}$ than $\mathbf{a_1,..,a_n}$ dependent:
$\mathbf{Ak}=\mathbf{0}$ with $k_i\neq 0$ and other $k_j=0$

Ex4 if $\mathbf{a_i}=\mathbf{a_j}$ than $\mathbf{a_1,..,a_n}$ dependent:
$\mathbf{Ak}=\mathbf{0}$ with $k_i=-k_j\neq 0$ and other $k_t=0$

4.2 Geometry and interpretation in 2-dimensions

Rank of a matrix r(**A**)= size of the largest nonsingular sub-matrix of **A**

r(**A**)≤ min(#columns, #rows)
A square:
if |**A**|=0 ↔ r(**A**)< #columns or rows

Ex5 **a**=2**b** depend., also |**A**|=0 r(**A**)=1 fig;
|**B**|=-14 and r(**B**)=2 figure
Ex6 |**A**|=0 singular, r(**A**)=2 |submatrix|≠0

Summary:
Ak=**0** $\mathbf{a}_i$ dependent →|**A**|=0 and **not all** $k_i=0$
Ak=**0** $\mathbf{a}_i$ independent →|**A**|≠0 and **all** $k_i=0$

4.3 Consistent set of linear equations

Ax=b <u>in</u>consistent $\leftrightarrow$ no solution for **x** and r(**A**)$\neq$r(**A|b**)

Ax=b consistent $\leftrightarrow$ r(**A**)=r(**A|b**)

Ex.p27 **A**(3x3) r(**A**)=2 r(**A|b**)=3 and |**A**|=0$\rightarrow$inconsistent$\rightarrow$no solutions

A square matrix

Ax=b consistent $\leftrightarrow$ r(**A**)=r(**A|b**)

- one solution $\leftrightarrow$ |**A**|$\neq$0; as many equations as unknowns
- more solutions $\leftrightarrow$ |**A**|=0; less equations than unknowns

Ex.p28 **A**(3x3), r(**A**)=r(**A|b**)=2$\rightarrow$consistent
|**A**|=0 $\rightarrow$many solutions

4.4 Examples - **Exercise**

$$\mathbf{A}=\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\\ 1 & 1 & 4\end{pmatrix} \qquad \mathbf{b}_1=\begin{pmatrix}1\\ 2\\ 0\end{pmatrix} \qquad \mathbf{b}_2=\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$$

$(\mathbf{A}|\mathbf{b}_1)$=(in)consistent?
$(\mathbf{A}|\mathbf{b}_2)$=(in)consistent?

Sove the equations $(\mathbf{A}|\mathbf{b}_1)=\mathbf{0}$ and $(\mathbf{A}|\mathbf{b}_2)=\mathbf{0}$.

Another way to solve the problem (costs more time)

- $r(\mathbf{A})$=?
  Check$|\mathbf{A}|$, check (2x2) submatr
- $r(\mathbf{A}|\mathbf{b}_1)$=?
  Check (3x3) submatr (no?); (2x2) submatr
- $r(\mathbf{A}|\mathbf{b}_2)$=?
  Check (3x3) submatrices

<u>Answer: **$\mathbf{Ax=b_1}$**</u>

$1x_1+2x_2+3x_3=1$ row1-2→ $-x_2+x_3=-1$

$1x_1+3x_2+2x_3=2$ row1-3→ $x_2-x_3= 1$

$1x_1+1x_2+4x_3=2$ ↑

Same equations

$x_2=x_3+1$substitute in 3

$x_1+5x_3=1$

One equation, 2 unknowns→many solutions

or

$|\mathbf{A}|=0$ $|\text{subm}|\neq 0$ →$r(\mathbf{A})=2$

$(\mathbf{A|b_1})$: 3 subm (3x3) all det=0

One (2x2)subm with det≠0→$r(\mathbf{A|b_1})=2$

$r(\mathbf{A})= r(\mathbf{A|b_1})=2$→ consistent, and

$|\mathbf{A}|=0$ →many solutions

<u>Answer **$\mathbf{Ax=b_2}$**:</u>

$1x_1+2x_2+3x_3=1$ row1-2→ $-x_2+x_3=-1$

$1x_1+3x_2+2x_3=2$ row1-3→ $x_2-x_3= 2$

$1x_1+1x_2+4x_3=3$ ↑

$x_2-x_3=2$ & $x_2-x_3=1$

no solution→ inconsistent

$r(\mathbf{A|b_2})=3$ as one (3x3)subm det≠0

$r(\mathbf{A}) \neq r(\mathbf{A|b_2})$ →inconsistent→no solution

4.5 Exercises

For next week: **1-5**

Solve the problems by solving the equations.
Use det and rank if more easy.

Remark (not in book!!!)

7 **Null-space**: solution **x** of **Ax=0**
Always solvable with **x=0,** thus
r(**A**)=r(**A|0**)

Ax=0 *consistent*

- |**A**|≠0 $\mathbf{a_1,..,a_n}$ independent, one solution for **x**, ie **x=0**: null-space **empty**
- |**A**|=0 $\mathbf{a_1,..,a_n}$ dependent, many solutions for **x** in the null-space

In general:
more unknowns than equations→many solutions→dependent set
one solution: zero→independent set

Try exercize 2 now.

2: We have to investigate the following set of equations:

$$\begin{pmatrix}1\\3\\2\end{pmatrix}a+\begin{pmatrix}2\\8\\-1\end{pmatrix}b+\begin{pmatrix}-1\\9\\2\end{pmatrix}c=\begin{pmatrix}6\\10\\-2\end{pmatrix}\rightarrow$$

$$a + 2b - c = 6 \qquad (1)$$
$$3a + 8b + 9c = 10 \qquad (2)$$
$$2a - b + 2c = -2 \qquad (3)$$

From (1) we get a=-2b+c+6. Plugging this into the remaining equations we obtain

$$b + 6c = -4 \qquad (2)$$
$$5b - 4c = 14 \qquad (3)$$

From (2) we get b=-6c-4. Plugging this into (3) we get c=-1. Solving this in (2) and (1) we get b=2 and a=1. Thus, the <u>vector is a linear combination of the set</u>.

Chapter 5
Eigenvalues and eigenvectors

5.1 Theory

$\mathbf{Ax}=\lambda\mathbf{x}$ (p×p)×(p×1)=p×1

$\mathbf{Ax}-\lambda\mathbf{x}=\mathbf{Ax}-\lambda\mathbf{Ix}=(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$

$\mathbf{A}-\lambda\mathbf{I}$: matrix **A** minus λ from diagonal

- det$(\mathbf{A}-\lambda\mathbf{I})\neq\mathbf{0}$
  $\mathbf{x}=\mathbf{0}$ the null-vector <u>trivial solution</u>
- det$(\mathbf{A}-\lambda\mathbf{I})=\mathbf{0}$
  non trivial solutions for **x**

det$(\mathbf{A}-\lambda\mathbf{I})=\mathbf{0}$ <u>characteristic function</u>

λ <u>eigenvalue of **A**</u>

x <u>eigenvector of **A**</u>

more solutions for λ: λ_i

more eigenvector **x**: $\mathbf{x}_i$

If $\mathbf{Ax}=\lambda\mathbf{x}$

than also $\mathbf{A}c\mathbf{x}=\lambda c\mathbf{x}$

standard $\mathbf{x}'\mathbf{x}=1$

Example

$$\mathbf{A} = \begin{pmatrix} 3 & 5 \\ -2 & -4 \end{pmatrix}$$

$$\mathbf{A}-\lambda\mathbf{I} = \begin{pmatrix} 3-\lambda & 5 \\ -2 & -4-\lambda \end{pmatrix}$$

characteristic function:

$|\mathbf{A}-\lambda\mathbf{I}|=(3-\lambda)(-4-\lambda)-5(-2)=\lambda^2+\lambda-2=0$

$\lambda=1 \rightarrow x_1=-5/2x_2$

$\lambda=-2 \rightarrow x_1=-x_2$

standard $\mathbf{x}'\mathbf{x}=1$

$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$

$$\begin{pmatrix} 3-\lambda & 5 \\ -2 & -4-\lambda \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\lambda=1$

$2x_1+5x_2=0 \rightarrow x_1=-5/2x_2$

$-2x_1-5x_2=0 \rightarrow$ idem

$\lambda=-2$

$5x_1+5x_2=0 \rightarrow x_1=-x_2$

$-2x_1-2x_2=0 \rightarrow$ idem

5.4 Properties of eigenvectors/values

A <u>symmetric</u> (p×p)

$$1: trace(\mathbf{A}) = \sum_{i=1}^{p} a_{ii} = \sum_{i=1}^{p} \lambda_i$$

$$2: \det(\mathbf{A}) = \prod_{i=1}^{p} \lambda_i$$

$$3: \mathbf{x}_i'\mathbf{x}_j = \sum_{k=1}^{p} x_{ki}x_{kj} = \begin{cases} 0 \; if \; i \neq j \\ 1 \; if \; i = j \end{cases}$$

4: collect $\mathbf{x}_i$ in **X** (p×p) and λ_i in Λ (diagonal, p×p)
AX= XΛ and **X′X= XX′=I (thus $\mathbf{X}^{-1}$=X′)**
XΛ column i multiplied with λ_i
(NOT ΛX row i multiplied with λ_i **)**

5: **A= XΛX′** and **X′X=I**

6: $\mathbf{A}^2 = \mathbf{X}\Lambda^2\mathbf{X}'$ and **X′X=I**
In general
$\mathbf{A}^k = \mathbf{X}\Lambda^k\mathbf{X}'$ and **X′X=I**

<u>Ad 1</u> **A**(p×p)

trace(**A**)=sum diagonal elements of **A:**

trace(**A**)=tr(**A**)=spoor(**A**)

$$tr(\mathbf{A}) = \sum_{i=1}^{p} a_{kk}$$

C(n×p), **B** (p×n), **CB**(n×n), **BC**(p×p)

tr(**CB**)=tr(**BC**)

Proof:

$\mathbf{CB}_{kk}$ diagonal elements of **CB**.

$$\mathbf{CB}_{kk} = \sum_{i=1}^{p} c_{ki} b_{ik} \rightarrow tr(\mathbf{CB}) = \sum_{k=1}^{n} \sum_{i=1}^{p} c_{ki} b_{ik}$$

$$\mathbf{BC}_{ii} = \sum_{k=1}^{n} b_{ik} c_{ki} \rightarrow tr(\mathbf{BC}) = \sum_{i=1}^{p} \sum_{k=1}^{n} c_{ki} b_{ik} \quad q.e.d.$$

Eigenvectors and eigenvalues of symmetric **A: X**(p×p) and Λ(p×p)

$$tr(\mathbf{A}) = tr(\Lambda)$$

Proof:

$$\mathbf{AX} = \mathbf{X}\Lambda \rightarrow \mathbf{AXX}^{-1} = \mathbf{A} = \mathbf{X}\Lambda\mathbf{X}^{-1}$$

$$tr(\mathbf{A}) = tr(\mathbf{X}\Lambda\mathbf{X}^{-1}) = tr(\Lambda\mathbf{X}^{-1}\mathbf{X}) = tr(\Lambda)$$

Ad 2

$$\det(\mathbf{A}) = \prod_{\mathbf{i=1}}^{\mathbf{p}} \lambda_i$$

Proof:

$$\mathbf{AX} = \mathbf{X}\Lambda$$

$$|\mathbf{AX}| = |\mathbf{A}||\mathbf{X}| = |\mathbf{X}||\Lambda|$$

$$|\mathbf{X}|\ \mathbf{scalar} \rightarrow |\mathbf{A}| = |\Lambda|$$

Ad 3

Inner product of **X**(p×p): **X′X**

If **X** are eigenvectors, than

$$\mathbf{X'X} = \mathbf{I} = \begin{pmatrix} 100......0 \\ 010......0 \\ .... \\ 000......1 \end{pmatrix} \ \textit{condition}$$

$$\textit{thus}\ \mathbf{x}_i'\mathbf{x}_j = \begin{cases} 1\ \textit{if}\ i = j \\ 0\ \textit{if}\ i \neq j \end{cases}$$

Ad 4

A(p×p) symmetric.

Solving $\mathbf{x}'\mathbf{A}\mathbf{x}$ maximal with $\mathbf{x}'\mathbf{x}=1$ gives: $\mathbf{A}\mathbf{x}=\mathbf{x}\lambda$ and $\mathbf{x}'\mathbf{x}=1$ (see appendix)

Collecting λ_i in Λ (diagonal) and $\mathbf{x}_i$ in **X** (p×p) and solving $\mathbf{X}'\mathbf{A}\mathbf{X}$ max with $\mathbf{X}'\mathbf{X}=I$ gives $\mathbf{A}\mathbf{X}=\mathbf{X}\Lambda$ and $\mathbf{X}'\mathbf{X}=I$

X and Λ are the eigenvectors and eigenvalues of **A**.

Ad 5

$\mathbf{A}$(p×p) symmetric and $\mathbf{X}$ (p×p)

If $\mathbf{AX}=\mathbf{X}\Lambda$ with $\mathbf{X'X}=\mathrm{I}$

than $\mathbf{X}^{-1}=\mathbf{X'}$

Proof:

$\mathbf{X'X}=\mathrm{I} \rightarrow \mathbf{X'XX}^{-1}=\mathbf{X'}\mathrm{I}=\mathrm{I}\mathbf{X}^{-1}$

Thus $\mathbf{X}^{-1}=\mathbf{X'}$ and $\mathbf{XX'}=\mathbf{I}$

Consequences:

$\mathbf{AX}=\mathbf{X}\Lambda \rightarrow \mathbf{AXX'}=\mathbf{A}=\mathbf{X}\Lambda\mathbf{X'}$

also

$\mathbf{AX}=\mathbf{X}\Lambda \rightarrow \mathbf{X'AX}=\mathbf{X'X}\Lambda=\Lambda$

Ad 6

$\mathbf{A}=\mathbf{X}\Lambda\mathbf{X}^{-1}$

$\mathbf{A}^{2}=\mathbf{AA}=\mathbf{X}\Lambda\mathbf{X}^{-1}\mathbf{X}\Lambda\mathbf{X}^{-1}=$

$=\mathbf{X}\Lambda\Lambda\mathbf{X}^{-1}=\mathbf{X}\Lambda^{2}\mathbf{X}^{-1}$

$\mathbf{A}^{k}=\mathbf{X}\Lambda^{k}\mathbf{X}^{-1}=\mathbf{X}\Lambda^{k}\mathbf{X'}$

5.5 Singular value decomposition

$\mathbf{B}(n \times p)$ with $n \geq p$:

$\mathbf{B}=\mathbf{K}\boldsymbol{\Lambda}\mathbf{L}'$ with $\mathbf{K}'\mathbf{K}=\mathbf{I}$ and $\mathbf{L}'\mathbf{L}=\mathbf{I}$

$\mathbf{K}$ $(n \times p)$, first p left eigenvectors of $\mathbf{B}$
$\mathbf{L}$ $(p \times p)$, p right eigenvectors of $\mathbf{B}$
$\boldsymbol{\Lambda}$ $(p \times p)$ first p singular values of $\mathbf{B}$
$\boldsymbol{\Lambda}$ diagonal

$\mathbf{B}'\mathbf{B} = \mathbf{L}\boldsymbol{\Lambda}^2\mathbf{L}'$ and $\mathbf{B}\mathbf{B}'=\mathbf{K}\boldsymbol{\Lambda}^2\mathbf{K}'$.

$\mathbf{K}$ $(n \times p)$, first p eigenvectors of $\mathbf{B}\mathbf{B}'$
$\mathbf{L}$ $(p \times p)$, p eigenvectors of $\mathbf{B}'\mathbf{B}$
$\boldsymbol{\Lambda}^2(p \times p)$ (first) p eigenvalues of $\mathbf{B}'\mathbf{B}$, $\mathbf{B}\mathbf{B}'$

Chapter 6
Application in statistics: MR and PCA

6.1 Multiple regression

y (n×1) criterion variable
X (n×m) columns: independent variables, predictors (first column: 1’s)
b (n×1) regression weights
e (n×1) vector of residuals (errors)

$\mathbf{y}=b_1+b_2\mathbf{x}_2+b_3\mathbf{x}_3+\mathbf{e}=\mathbf{Xb+e}$

$$\mathbf{X}=\left(\mathbf{1}\,|\,\mathbf{x_2},\mathbf{x_3}\right)$$

$x_1'=(1,1,\ldots.,1)$

$\mathbf{e=y\text{-}Xb}$ and $\mathbf{e'e}$ minimal (SSE)
thus
f= $\mathbf{e'e=(y\text{-}Xb)'(y\text{-}Xb)}$ minimal
The derivative of f with respect to **b** must be 0.

f= $\mathbf{y'y\text{-}2y'Xb+\ bX'Xb}$ minimal

Derivatives with respect to **b** of **a′b** and **b′Ab (A** symmetric):

$$\frac{\partial \mathbf{a}'\mathbf{b}}{\partial \mathbf{b}} = \mathbf{a} \quad \textit{and} \quad \frac{\partial \mathbf{b}'\mathbf{A}\mathbf{b}}{\partial \mathbf{b}} = 2\mathbf{A}\mathbf{b}$$

$$\frac{\partial \mathbf{y}'\mathbf{X}\mathbf{b}}{\partial \mathbf{b}} = \mathbf{X}'\mathbf{y}$$

$$\frac{\partial \mathbf{b}'\mathbf{X}'\mathbf{X}\mathbf{b}}{\partial \mathbf{b}} = 2\mathbf{X}'\mathbf{X}\mathbf{b}$$

f**= e′e =y′y-2y′Xb+ b′X′Xb** minimal

$$\frac{\partial f}{\partial \mathbf{b}} = -2\mathbf{X}'\mathbf{y} + 2\mathbf{X}'\mathbf{X}\mathbf{b} = 0$$

$$2\mathbf{X}'\mathbf{X}\mathbf{b} = 2\mathbf{X}'\mathbf{y}$$

$$\hat{\mathbf{b}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

$$\hat{\mathbf{y}} = \mathbf{X}\hat{\mathbf{b}}$$

6.3 Principal Component Analysis

X data (n×p), p variables (means 0)
y= weighted sum of **X**: **Xb**

y=Xb (**y** *<u>unknown</u>*) and
optimize f**=y′y= b′X′Xb** with **b′b=1**

optimize f***=b′X′Xb+**λ**(b′b-1)**
λ Lagrange multiplier (λ≠0)

$$\frac{\partial f^*}{\partial \mathbf{b}} = 2\mathbf{X}'\mathbf{X}\mathbf{b} - 2\lambda\mathbf{b} = 0$$

$$\frac{\partial f^*}{\partial \lambda} = \mathbf{b}'\mathbf{b} - 1 = 0$$

$$\mathbf{X}'\mathbf{X}\mathbf{b} = \lambda\mathbf{b} \;\; \textit{with}\; \mathbf{b}'\mathbf{b} = 1$$

b eigenvectors of **X′X**
λ eigenvalues of **X′X**
and
X′XB=BΛ with **B′B=I**

If **X** is in deviation from column means (**X′u**=**0**), then **y=Xb** also in deviation from mean, as **y′u=b′X′u** =**b′0=**0
X′X/n is the covariance matrix of **X.**

If **X** is standardized: **(X′u**=**0** and diag(**X′X**)**=**n**I),** then **X′X**/n is the correlation matrix of **X.**

Maximising **b′X′Xb,** is maximising the covariance of the **y** vectors, or the linear combination of the **X**-vectors.

The solution is

$$\mathbf{X}'\mathbf{X}\mathbf{b} = \lambda\mathbf{b} \;\; with \; \mathbf{b}'\mathbf{b} = 1,$$

or

$$\frac{1}{n}\mathbf{X}'\mathbf{X}\mathbf{b} = \frac{\lambda}{n}\mathbf{b} \;\; with \; \mathbf{b}'\mathbf{b} = 1,$$

Thus eigenvectors and eigenvalues/n of **X′X** are characteristics to the covariance or correlation matrix.

.

Appendix
Derivatives of vectors and matrices

1 f=**a′x=** $a_1x_1, \ldots, a_nx_n$

$$\frac{\partial f}{\partial x_i} = a_i$$

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \mathbf{a}$$

$$\frac{\partial f}{\partial \mathbf{x}} = \frac{\partial \mathbf{a}'\mathbf{x}}{\partial \mathbf{x}} = \mathbf{a}$$

2 f=**x′Ax** **A** <u>symmetric</u>

$$\frac{\partial f}{\partial \mathbf{x}} = \frac{\partial \mathbf{x}'\mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x} \quad see\ p.47\ no\ 2$$

3 optimize

$f = \mathbf{x}' \mathbf{A}\mathbf{x}$ with $\mathbf{x}'\mathbf{x} = \mathbf{1}$

or

$f^* = \mathbf{x}' \mathbf{A}\mathbf{x} + \lambda(\mathbf{x}'\mathbf{x} - \mathbf{1})$

λ Lagrange multiplier ($\lambda \neq 0$)

$$\frac{\partial f^*}{\partial \mathbf{x}} = 2\mathbf{A}\mathbf{x} - 2\lambda\mathbf{x} = 0 \rightarrow \mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$\frac{\partial f^*}{\partial \lambda} = \mathbf{x}'\mathbf{x} - 1 = 0 \rightarrow \mathbf{x}'\mathbf{x} = \mathbf{1}$$

$\mathbf{x}$ eigenvectors of $\mathbf{A}$

λ eigenvalues of $\mathbf{A}$

and

$\mathbf{A}\mathbf{X} = \mathbf{X}\Lambda$ with $\mathbf{X}'\mathbf{X} = \mathbf{I}$

Optimum $f = \mathbf{x}_1'\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1'\mathbf{x}_1 = \lambda_1$

4 **x** and **y** vectors

θ angle between vector **x** and **y**

$$\cos\theta = \frac{\sum_i x_i y_i}{\sqrt{\sum_i x_i^2}\sqrt{\sum_i y_i^2}} = \frac{\mathbf{x}'\mathbf{y}}{\sqrt{\mathbf{x}'\mathbf{x}}\sqrt{\mathbf{y}'\mathbf{y}}}.$$

$$\sqrt{\mathbf{y}'\mathbf{y}} = \|\mathbf{y}\| = \mathit{length}\ \mathbf{y}$$

Remark:

If **x** and **y** are vectors with mean zero ($\mathbf{u}'\mathbf{x}=0$, $\mathbf{u}'\mathbf{y}=0$):

$$\cos\theta = cor(\mathbf{x},\mathbf{y}) \frac{\frac{\mathbf{x}'\mathbf{y}}{n}}{\sqrt{\frac{\mathbf{x}'\mathbf{x}}{n}}\sqrt{\frac{\mathbf{y}'\mathbf{y}}{n}}}$$

If **x** and **y** are vectors with mean zero and variance 1 ($\mathbf{x}'\mathbf{x}/n=1$, $\mathbf{y}'\mathbf{y}/n=1$) then

$$\cos\theta = \mathrm{cor}(\mathbf{x},\mathbf{y}) = \mathbf{x}'\mathbf{y}/n.$$

Exercise Appendix

1**:**Maximise f=**b′Ab** under the condition that **b′b=1**, where

$$\mathbf{A} = \begin{pmatrix} \mathbf{2} & \mathbf{6} \\ \mathbf{1} & \mathbf{1} \end{pmatrix},$$

and solve **b.** What is the optimum of f?

Matrix Algebra, K. Namboodiri

Ch 1 Introduction

Arrays (matrices)

- Addition
- Subtraction
- Vectors
  1 Linear equations
  2 Inner products
- Multiplication of vectors and matrices
- Identity matrix

Ch 2 Elementary operations and inverse

- Elementary operations
- Echelon matrices
- Inverse of a square matrix
- Calculation of the inverse
- Inverse and the solution of a system of equations
- Applications MR and Markov chains

Ch 2

Elementary row operations

1. interchange 2 rows
2. multiply a row by a scalar
3. adding a non-zero multiple of one row to another row

p.28: $\mathbf{A}, \mathbf{E_1A}, \mathbf{E_2\,E_1A}$

p.29,30: $\mathbf{A}, \mathbf{E_1A}, \mathbf{E_2A}, \mathbf{E_3A}$

p.30: Postmultiplications with changes columns

p.31: ***Echelon matrices***

- row 1 to k one or more nonzero elements
- first nonzero element from left to right is 1
- first nonzero element of row i to the right of the first nonzero element of row i-1
- after k-th row – zero rows

ex p 32 $\mathbf{A_1=E_1A_0}$, etc

Echelon matrix obtained by elementary row operations

Ex p 33 $\mathbf{B_1=E_1A_0}$, etc

Inverse

B: inverse(**A**):

BA=AB=I

$\mathbf{B}=\mathbf{A}^{-1}$

How to compute the inverse:
by Echelon matrix and than changing this matrix into **I.**

Ex p.37: $\mathbf{G}, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{E}_4$

$\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{G}=\mathbf{I}$, and $\mathbf{G}^{-1}=\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1$

Ch 3 workbook

Linear equations

$\mathbf{Ax}=\mathbf{b}$, solve $\mathbf{x}$, than $\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$
by elementary row operations

Direct

$$\mathbf{A}^{-1}=\frac{1}{|\mathbf{A}|}(\mathbf{A}^{c})'$$

$\mathbf{A}^c$ = matrix with **cofactors**

cofactor $c_{ij}=(-1)^{i+j}\times$ minor(i,j)

minor(i,j)=det{A if r_i, c_j deleted}

p.40 **A** and $\mathbf{A}^{-1}$

Example p.40

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & 9 \end{pmatrix}$$

$$\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 9 \end{pmatrix}$$

$$\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = Echelon$$

$$\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{E}_5\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{E}_6\mathbf{E}_5\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

$$\mathbf{E_2E_1}$$
$$\mathbf{E_3E_2E_1}$$
$$\mathbf{E_4E_3E_2E_1}$$
$$\mathbf{E_5E_4E_3E_2E_1}$$
$$\mathbf{E_6E_5E_4E_3E_2E_1=A^{-1}}$$

$$\mathbf{E_2E_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$\mathbf{E_3E_2E_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\mathbf{E_4E_3E_2E_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\mathbf{E_5E_4E_3E_2E_1} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -6 & 4 \\ 1 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\mathbf{E_6E_5E_4E_3E_2E_1} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & -6 & 4 \\ 1 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix} = \mathbf{A}^{-1}$$

$$\mathbf{A}^{-1}\mathbf{A} = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Regression

$\mathbf{y=Xb+e}$

$\mathbf{X=(u|x)}$ **with** $\mathbf{u'u=}n$

$\mathbf{y=ub_0+xb_1+e}$

$\mathbf{(y-ub_0-xb_1)'(y-ub_0-xb_1)}=$

$=\mathbf{(y-Xb)'(y-Xb)}$ minimal

This gives: $\mathbf{X'Xb=X'y}$

$$\mathbf{X}^T\mathbf{Xb}=\begin{pmatrix}\mathbf{u}'\mathbf{u} & \mathbf{u}'\mathbf{x}\\ \mathbf{u}'\mathbf{x} & \mathbf{x}'\mathbf{x}\end{pmatrix}\times\begin{pmatrix}b_0\\ b_1\end{pmatrix}=\begin{pmatrix}\mathbf{u}'\mathbf{y}\\ \mathbf{x}'\mathbf{y}\end{pmatrix} \quad [2.14]$$

p.43 compute

$\mathbf{u'u=}n=5$

$\mathbf{u'x=}\Sigma x_i=265$

$\mathbf{x'x=}\Sigma x_i^2=14199$

$\mathbf{u'y=}\Sigma y_i=293$

$\mathbf{x'y=}\Sigma x_i y_i=15696$

$\mathbf{b=(X'X)^{-1}X'y}$

Application input-output analysis ex p 48

$\mathbf{x=Ax+d}$

$\mathbf{(I-A)x=d}$

$\mathbf{x=(I-A)^{-1}d}$

Exercises chapter 1 and 2 Namboodiri

1 Inverse

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} \quad and \quad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

Ax=b

- Solve **x** by computing $\mathbf{A}^{-1}$
  with $\mathbf{A}^{-1}=\{1/|A|\}(A^c)'$
  Cheque $\mathbf{A}^{-1}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{-1}$
- Compute $\mathbf{A}^{-1}\mathbf{b}$
- Compute **x** by solving the equations
- Solve **x** by means of the Echelon matrix and then $\mathbf{A}^{-1}$ from $E_k \times \ldots \times E_1$

2 Multiple regression

$$\mathbf{X} = \begin{pmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 4 \\ 1 & 3 \\ 1 & 4 \\ 1 & 3 \\ 1 & 3 \end{pmatrix} \quad and \quad \mathbf{y} = \begin{pmatrix} 2 \\ 6 \\ 5 \\ 3 \\ 5 \\ 4 \\ 2 \end{pmatrix}$$

Y=Xb+e

Compute $\hat{\mathbf{b}}$ and $\mathbf{X}\hat{\mathbf{b}}$.

Compute $\mathbf{y} - \hat{\mathbf{y}}$.

Compute SST=SStotal, SSE=SSresiduals, SSM=SSregression and the df.

Compute F.

Ch3 Namboodiri

Simultaneous linear equations

Linear dependence among a set of vectors

Definition 1

A set of vectors is linearly dependent if at least one vector can be expressed as a linear combination of the other vectors.

Definition 2

A set of vectors, $\mathbf{a_1},..,\mathbf{a_n}$ is linearly dependent if there exist scalars $k_1,..,k_n$, not all zero, such that $k_1\mathbf{a_1}+..+ k_n\mathbf{a_n} = \mathbf{0}$

$\mathbf{A}=\{\mathbf{a_1},..,\mathbf{a_n}\}$ **and** $\mathbf{k}'=(k_1,..,k_n)$

Set **A** linearly *dependent*$\rightarrow$**Ak=0** with at least one $k_i \neq 0$.

Set **A** linearly *independent*$\rightarrow$**Ak=0** with all $k_i=0$.

E_i matrices with elementary row operations

$E_k..E_1A$ Echelon form:

If one or more rows equal to zero: rows of set **A** linearly dependent.

$$\mathbf{A} = \begin{pmatrix} 3 & 5 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$$

$$\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 5 \\ 2 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 5/3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix}$$

$$\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 5/3 \\ 2 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 5/3 \\ 0 & -7/3 \\ 0 & 7 \end{pmatrix}$$

$$\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/7 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 5/3 \\ 0 & -7/3 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 5/3 \\ 0 & 1 \\ 0 & 7 \end{pmatrix}$$

$$\mathbf{E}_4\mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 5/3 \\ 0 & 1 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 5/3 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Echelon form, Table 4.3 p.52

Row 3 zero→rows linearly dependent

p.52 columns **A'** (in)dependent)?

$$\mathbf{A}\mathbf{k}=\mathbf{0}=\begin{pmatrix}3 & 2 & -3\\ 5 & 1 & 2\end{pmatrix}\mathbf{k}$$

$3k_1+2k_2-3k_3=0$ (1)
$5k_1+1k_2+2k_3=0$ (2)
r1-2r2=$-7k_1-7k_3=0 \rightarrow k_3=-k_1$ in (1) or (2)
$\rightarrow k_2=-3k_1$
e.g. $k_1=1 \rightarrow k_2=-3$ $k_3=-1$ satisfies
or $k_1=2$ etc.
3 unknowns and 2 equations, many solutions, thus columns of **A**′ dependent, there is a linear combination with nonzero elements.

Book:
$\mathbf{E_4E_3E_2E_1A}$ row3=[0 0] p.53
By making an Echelon matrix of **A**, the number of nonzero rows is the rank of **A**

Workbook: One (2x2) submatrix of **A** has det≠0→r(**A**)=2

p.54

$$\mathbf{Ex1: A} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$$

Dependent columns?

Det(A)=-1 independent

Echelon: no zero rows, r(A)=2

$$\mathbf{Ex2: A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Det(A)=0 dependent

Echelon: one zero row, r(A)=1

$$\mathbf{Ex3: A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}$$

One 2x2 submatrix det≠0, r(A)=2

Echelon: no zero rows, r(A)=2

$$\mathbf{Ex4: A} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{pmatrix}$$

One 2x2 submatrix has det≠0, r(A)=2
Echelon: 2 zero rows of 4, r(A)=2

Def p.55
Echelon matrix of **A** by row (column) operations: the no. of nonzero rows (columns) is rank(**A**).
If **A**(nxm): r(A) ≤ min(n,m).

A(nxn) square
A *full rank* $\leftrightarrow$ r(**A**)=n and |**A**|≠0.
If **A** of full rank, **A** has an inverse and **A** is called *nonsingular*

Simulatneous linear equations

Ax=**b**,

no solution for **x**, one solution or many solutions.

$\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$ if $\mathbf{A}^{-1}$ exists.

Ax=**b**

Augmented matrix: (**A|b**)

$$\mathbf{A}=\begin{pmatrix}3 & 1\\ 1 & 1\end{pmatrix},\ \mathbf{b}=\begin{pmatrix}7\\ 3\end{pmatrix}$$

$$(\mathbf{A}|\mathbf{b})=\begin{pmatrix}3 & 1 & 7\\ 1 & 1 & 3\end{pmatrix}$$

r(**A**) and r(**A|b**)

If r(**A**)≠r(**A|b**) inconsistent $\leftrightarrow$ no solution

If r(**A**)=r(**A|b**) consistent

- det(**A**)≠0$\leftrightarrow$one solution
- det(**A**)=0$\leftrightarrow$more solutions

p.58

<u>Full rank case</u>

Echelon matrix multiplied such that I appears

$\mathbf{A^{-1}=E_k\ldots\ldots E_1}$

<u>Less than full rank case</u>

Table 3.3 p.60

1 3x+1y=7

2 6x+2y=14

$$\mathbf{(A\,|\,b)} = \begin{pmatrix} 3 & 1 & 7 \\ 6 & 2 & 14 \end{pmatrix}$$

$$\mathbf{E_1(A|b)} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 1 & 7 \\ 6 & 2 & 14 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{3} & \frac{7}{3} \\ 6 & 2 & 14 \end{pmatrix}$$

$$\mathbf{E_2E_1(A|b)} = \begin{pmatrix} 1 & 0 \\ -6 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & \frac{1}{3} & \frac{7}{3} \\ 6 & 2 & 14 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{3} & \frac{7}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

x+(1/3)y=7/3→many solutions

Also r(A|b)=r(A)=1+det(A)=0

$$\textbf{p.61, Table 3.4}: \ (\mathbf{A}\,|\,\mathbf{b}) = \begin{pmatrix} 1 & 1 & 1 & 5 \\ 1 & 2 & -1 & 0 \\ 2 & 3 & 0 & 5 \end{pmatrix}$$

<u>Remark</u>: r3=r1+r2 dependence

r2-r1; r3-2r1; r3-r2; r1-2r2

$$\mathbf{E_4 E_3 E_2 E_1 (A\,|\,b)} = \begin{pmatrix} 1 & 0 & 3 & 10 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2 eq, 3 unknowns, many solutions

Also: r(**A|b**)=r(**A**)=2 consistent,
det|A|=0→many solutions

Thus **A-**vectors linear combination of **b**.

p.63-64

Solving simultaneous liear equatuions

- n equations and r(**A**)<n, than there are n-r(A) redundant equations, delete them.
- Parameterize n-r(A) unknows, and rewrite the retained r(A)equations in r(A) unknowns, and solve by the method of solving the full rank case.

P.64 ex [3.17]
(x,y,z) unknows, 3 equations, r(A)=2 by Echelon matrix. Delete equation 3, and parameterize 1 unknown e.g. z=θ, and substitute θ in the other equations.
Full rank problem, solution in [3.19].
Add z=θ [3.20]. Thus many solutions.

Ax=**b and** |**A**|=0
x=**Gb** with **G** *<u>generalized inverse</u>* of **A**

G contains the inverse of the
smallerA , with r(**smallerA**)=r(**A**) and zero row(s) and column(s).
See [3.22] and the lowest row of p.65

p.67 below
SmallerA many possibilities, all different **G**, thus many solutions.

<u>Generalized inverse</u> of A: A^-

<u>Not for examination</u>

General solution to **Ax=b**

$\mathbf{x}=\mathbf{A}^-\mathbf{b}+(\mathbf{I}-\mathbf{A}^-\mathbf{A})\theta$ see [3.30]

<u>Definition</u>

$\mathbf{A}^-$ is the generalized inverse of **A** if $\mathbf{A}\mathbf{A}^-\mathbf{A}=\mathbf{A}$ holds.

Homogeneous equations p.70

Ax=0

x=0 satisfies always: trivial solution

Echelon form

m linear equations and n unknowns (variables) and m>n: **A**(m×n).

If r(**A**)=n → **x**=**0** the only solution p.71

If r(**A**)<n → many solutions p.72

p.74

Eigenvalues and Eigenvectors

Definition det and inverse *workbook*

A *singular* if $|\mathbf{A}|=0$

$\mathbf{Ap}=\mathbf{p}\lambda$ and $\mathbf{p}'\mathbf{p}=1$

λ eigenvalue

p eigenvector

$(\mathbf{A}-\lambda\mathbf{I})\mathbf{p}=\mathbf{0}$. The solution $\mathbf{p}=\mathbf{0}$ is trivial,

$\rightarrow(\mathbf{A}-\lambda\mathbf{I})$ singular

$\rightarrow|\mathbf{A}-\lambda\mathbf{I}|=\mathbf{0}$ (characteristic equation)

gives values for λ. Next $(\mathbf{A}-\lambda\mathbf{I})\mathbf{p}=\mathbf{0}$ can be solved.

In general

$\mathbf{AP}=\mathbf{P}\Lambda$ and $\mathbf{P}'\mathbf{P}=\mathbf{I}$

$\mathbf{A}=\mathbf{P}\Lambda\mathbf{P}^{-1}$

PCA

Table 4.1 p.89

$\mathbf{W}$=matrix of Sum of Squares of variables (in deviation for mean)

$|\mathbf{W}-\lambda\mathbf{I}|=\mathbf{0}$ solve λ [4.27], 2 solutions

$(\mathbf{W}-\lambda\mathbf{I})\mathbf{c}=\mathbf{0}$ solve $\mathbf{c}$ [4.30], 2 solutions

λ eigenvalues of $\mathbf{W}$

$\mathbf{c}$ eigenvectors of $\mathbf{W}$

PCA: *workbook*

Model $\mathbf{Y}=\mathbf{X}\mathbf{b}$

$\mathbf{X}'\mathbf{X}\mathbf{b}=\mathbf{b}\lambda$ with $\mathbf{b}'\mathbf{b}=\mathbf{1}$

define $\mathbf{W}=\mathbf{X}'\mathbf{X}$ then $\mathbf{W}\mathbf{b}=\mathbf{b}\lambda$

λ eigenvalue, $\mathbf{b}$ eigenvector of $\mathbf{X}'\mathbf{X}$ **or** $\mathbf{W}$

In general

$\mathbf{X}'\mathbf{X}\mathbf{B}=\mathbf{B}\Lambda$ with $\mathbf{B}'\mathbf{B}=\mathbf{1}$

Model [4.34] $\mathbf{Y}=\mathbf{X}\mathbf{B}+\mathbf{E}$ Factor analysis

p.93

If $\mathbf{W}$ symmetric and

$\mathbf{WP}=\mathbf{P}\boldsymbol{\Lambda}$ with $\mathbf{P}'\mathbf{P}=\mathbf{I}$, than $\mathbf{P}'=\mathbf{P}^{-1}$

$\mathbf{P}'\mathbf{P}=\mathbf{PP}'=\mathbf{I}$, **P** *orthogonal.*

$\mathbf{W}=\mathbf{P}\boldsymbol{\Lambda}\mathbf{P}'$

If $\mathbf{W}$ symmetric and $\mathbf{W}=\mathbf{X}'\mathbf{X} \rightarrow$ all eigenvalues are positive